Setting Limits in the Presence of Nuisance Parameters

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The Problem

- \( x \) events in the signal region
- \( y \) events in data sidebands (or from MC), measured with some uncertainty, statistical and systematic
- \( z \) a measurement of the efficiency, measured with some uncertainty, statistical and systematic

→ How do we set limits on the signal rate?
Previous Solution: Cousins-Highland

- Basically, integrate out the nuisance parameter.
- Problem 1: "hidden" Bayesian method – what should be used as a prior?
- Problem 2: does it work, i.e. does it have coverage? So far nobody knows, although we might soon
New Solution – Profile Likelihood

Need some notation:

• $x$ – number of events in signal region
• $y$ – number of events in data sidebands
• $\tau$ – relative “size” of background region to signal region, so that $y/\tau$ is estimate of background in signal region
• m – number of MC events to test efficiency
• z – number of MC events that survive the cuts

So \( z/m \) is an estimate of the efficiency

Unknown Parameters:
• \( \mu \) – signal rate (what we want to know)
• \( b \) – background rate in signal region
• \( e \) – efficiency
Probability Model:
\( X \sim \text{Pois}(e\mu+b), \ Y \sim \text{Pois}(\tau b), \ Z \sim \text{Binom}(m,e) \)

Loglikelihood:

\[
\ell(\mu,b,e) = (-2) \times (x \log(e\mu+b) - \log(x!) - (e\mu+b) + y \log(\tau b) - \log(y!) - \tau b + \log(m!) - \log(z!) - \log((m-z)!) + z \log(e) + (m-z) \log(1-e))
\]

is a function of all parameters

Idea: for each \( \mu \) find \( b \) and \( e \) which make the observations most likely – profile likelihood
Illustration of Profile Likelihood

- Case: $x=8$
- $y=15$
- $\tau=5.0$
- $e=100\%$ (known)
- $\mu$ fixed at 2
  → $bhat = 3.33$
Sometimes this can be done analytically, sometimes (like here) it has to be done numerically.

Result: given the data \((x,y,z,\tau,m)\) the profile likelihood is a function of \(\mu\) alone

\[ \rightarrow \text{no more nuisance parameters} \]
One Problem: $x < \frac{y}{\tau}$

- Then mle of $\mu < 0$
- Example: same as before, but $x=2$, so $x - \frac{y}{\tau} = -1.0$
- 90% upper limit is 2.45
Even worse:

- Same as last, but \( y = 35.0 \)
- So we expect 7 events just from background, but we only see 2

Note: even if \( \mu = 0 \) this happens only about 5% of the time.
Two ways to handle this:

• keep $y, z, \tau, m$ fixed, find smallest $x$ for which upper limit is greater than 0
  → intuitive meaning of “upper limit”
  “unbounded likelihood method”

• use constrained likelihood, i.e. require $\text{mle} \geq 0$ always
  → uses physical limits on parameters
  “bounded likelihood method”
Method can deal with other situations:

- Background and/or Efficiency are known without error

- Background is Gaussian instead of Poisson:
  \[ y \sim N(b, \sigma_b) \]

- Efficiency is Gaussian instead of Binomial:
  \[ z \sim N(e, \sigma_e) \]

→ Allows incorporation of systematic errors
So, does it work?

- Confidence Intervals work if they have coverage:
  - Fix $\mu$, $b$, $e$, $sdb$, $sde$ and $\alpha$
  - Generate $y_1,.., y_n \sim N(b, sdb)$
  - Generate $z_1,.., z_n \sim N(e, sde)$
  - Generate $x_1,.., x_n \sim \text{Pois}(e\mu+ b)$
  - Find $(1-\alpha)100\%$ CI’s $(L_i,U_i)$ for $i=1,..,n$
  - Find percentage $p$ with $L_i \leq \mu \leq U_i$
  - If $p \geq (1-\alpha)100\%$, we have correct coverage
  - Repeat for many values of $\mu$, $b$, $e$, $sdb$, $sde$ and $\alpha$
Example

- Background – Gaussian with error 0.5
- Efficiency – Gaussian with mean 0.85 and error 0.075
- Signal rate varies from 0 to 10 in steps of 0.1
- Background rate varies from 0 to 10 in steps of 2
- Nominal coverage rate 90%
- Orange – unbounded likelihood
- Blue - bounded likelihood
Features of our Method:

• Always yields positive upper limit
• Smooth transition from upper limits to two-sided intervals
• Now available as part of ROOT: TRolke
• Limits are consistent as errors on nuisance parameters become small:
TRolke Intervals

Use instead of Feldman Cousins?
Isn’t it a marvelous new method?

• See F. James, MINUIT Reference Manual, CERN Library Long Writeup D506, p.5: "The MINOS error for a given parameter is defined as the change in the value of the parameter that causes $F'$ to increase by the amount UP, where $F'$ is the minimum w.r.t to all other free parameters."

NO! It is a marvelous old method ... nobody knew how marvelous though.

Confidence Interval

Profile Likelihood (in $X^2$ approximation)

$\Delta X^2 = 2.71$ (90%), $\Delta X^2 = 1.07$ (70%)
Summary

• Profile Likelihood is a general technique for dealing with nuisance parameters
• It is familiar to physicists as part of MINUIT
• For the problem of setting limits for rare decays it yields a method with good coverage and some nice properties
• It is available as part of ROOT
• The End